

Algebraic setup of non-strict multiple zeta values

Shuichi Muneta

1 Introduction

The multiple zeta values and non-strict multiple zeta values (MZVs and NMZVs, for short) are defined respectively by

$$\zeta(k_1, k_2, \dots, k_n) := \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}},$$

$$\bar{\zeta}(k_1, k_2, \dots, k_n) := \sum_{m_1 \geq m_2 \geq \dots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}},$$

where k_1, k_2, \dots, k_n are positive integers and $k_1 \geq 2$. Considerable amount of work on MZVs has been done from various aspects and interests.

The MZVs have many relations among them (duality formula, sum formula, Hoffman's relations, Ohno's relations, derivation relations and cyclic sum relations, cf. [H1], [HO], [IKZ], [O]) and these relations can be described in purely algebraic manner (cf. [IKZ]). On the other hand, NMZVs have not been investigated so much compared to MZVs. But recently, a few works on NMZVs have appeared ([AO], [OW]) and they indicate that NMZVs possess similar properties to MZVs.

In this article, we introduce an algebraic setup of NMZVs and prove some relations of NMZVs, which are analogous to Hoffman's relations of MZVs, by using this algebraic setup of NMZVs.

2000 *Mathematics Subject Classification.* Primary 11M41.

2 Algebraic setup of NMZVs

2.1 Algebraic setup of MZVs

We summarize the algebraic setup of MZVs introduced by Hoffman (cf. [H2], [IKZ]). Let $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$ be the noncommutative polynomial ring in two indeterminates x, y and \mathfrak{H}^1 and \mathfrak{H}^0 its subrings $\mathbb{Q} + \mathfrak{H}y$ and $\mathbb{Q} + x\mathfrak{H}y$. We set $z_k = x^{k-1}y$ ($k = 1, 2, 3, \dots$). Then \mathfrak{H}^1 is freely generated by $\{z_k\}_{k \geq 1}$. For any word w , let $l(w)$ be the degree of w with respect to y , and $|w|$ the total degree.

We define the \mathbb{Q} -linear map (called evaluation map) $Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ by

$$Z(1) = 1 \text{ and } Z(z_{k_1} z_{k_2} \cdots z_{k_n}) = \zeta(k_1, k_2, \dots, k_n).$$

We next define two products of MZVs. The one is the harmonic product $*$ on \mathfrak{H}^1 defined by

$$\begin{aligned} 1 * w &= w * 1 = w, \\ z_k w_1 * z_l w_2 &= z_k (w_1 * z_l w_2) + z_l (z_k w_1 * w_2) + z_{k+l} (w_1 * w_2) \end{aligned}$$

($k, l \in \mathbb{Z}_{\geq 1}$ and w, w_1, w_2 are words in \mathfrak{H}^1), together with \mathbb{Q} -bilinearity. The harmonic product $*$ is commutative and associative, therefore \mathfrak{H}^1 is \mathbb{Q} -commutative algebra with respect to $*$. We denote it by \mathfrak{H}_*^1 . The subset \mathfrak{H}^0 is a subalgebra of \mathfrak{H}^1 with respect to $*$ and we denote it by \mathfrak{H}_*^0 . We then have

$$Z(w_1 * w_2) = Z(w_1)Z(w_2)$$

for any $w_1, w_2 \in \mathfrak{H}^0$. The other product is the shuffle product \mathfrak{m} on \mathfrak{H} defined by

$$\begin{aligned} 1 \mathfrak{m} w &= w \mathfrak{m} 1 = w, \\ u_1 w_1 \mathfrak{m} u_2 w_2 &= u_1 (w_1 \mathfrak{m} u_2 w_2) + u_2 (u_1 w_1 \mathfrak{m} w_2) \end{aligned}$$

($u_1, u_2 \in \{x, y\}$ and w, w_1, w_2 are words in \mathfrak{H}), together with \mathbb{Q} -bilinearity. The shuffle product \mathfrak{m} is also commutative and associative, therefore \mathfrak{H} is \mathbb{Q} -commutative algebra with respect to \mathfrak{m} . We denote it by $\mathfrak{H}_{\mathfrak{m}}$. The subsets \mathfrak{H}^1 and \mathfrak{H}^0 are subalgebras of \mathfrak{H} with respect to \mathfrak{m} and we denote them by $\mathfrak{H}_{\mathfrak{m}}^1, \mathfrak{H}_{\mathfrak{m}}^0$ respectively. For this product, we also have

$$Z(w_1 \mathfrak{m} w_2) = Z(w_1)Z(w_2)$$

for any $w_1, w_2 \in \mathfrak{H}^0$.

The finite double shuffle relations of MZVs is then

$$Z(w_1 * w_2 - w_1 \amalg w_2) = 0 \quad (w_1, w_2 \in \mathfrak{H}^0).$$

The evaluation map is generalized by the following proposition to get the extended double shuffle relations of MZVs.

Proposition 2.1 ([IKZ]). *We have two algebra homomorphisms*

$$Z^* : \mathfrak{H}_*^1 \longrightarrow \mathbb{R}[T] \quad \text{and} \quad Z^\amalg : \mathfrak{H}_\amalg^1 \longrightarrow \mathbb{R}[T]$$

which are uniquely characterized by the properties that they both extend the evaluation map $Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ and send y to T .

Then we have the extended double shuffle relations of MZVs.

Theorem 2.2 ([IKZ]). *For any $w_1 \in \mathfrak{H}^1$ and $w_2 \in \mathfrak{H}^0$, we have*

$$Z^*(w_1 \amalg w_2 - w_1 * w_2) = 0 \quad \text{and} \quad Z^\amalg(w_1 \amalg w_2 - w_1 * w_2) = 0.$$

2.2 Algebraic setup of NMZVs

In this subsection, we introduce the algebraic setup of NMZVs. Define \mathbb{Q} -linear map $\overline{Z} : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ by

$$\overline{Z}(1) = 1 \quad \text{and} \quad \overline{Z}(z_{k_1} z_{k_2} \cdots z_{k_n}) = \overline{\zeta}(k_1, k_2, \dots, k_n).$$

We call this map n -evaluation map. We next define the n -harmonic product $\overline{*}$ on \mathfrak{H}^1 , which is the NMZV-counterpart of the harmonic product $*$, inductively by

$$\begin{aligned} 1 \overline{*} w &= w \overline{*} 1 = w, \\ z_k w_1 \overline{*} z_l w_2 &= z_k (w_1 \overline{*} z_l w_2) + z_l (z_k w_1 \overline{*} w_2) - z_{k+l} (w_1 \overline{*} w_2) \end{aligned}$$

($k, l \in \mathbb{Z}_{\geq 1}$ and w, w_1, w_2 are words in \mathfrak{H}^1), together with \mathbb{Q} -bilinearity. The n -harmonic product $\overline{*}$ has the following properties.

Proposition 2.3. *The n -harmonic product $\overline{*}$ is commutative and associative.*

Proof. We can prove the assertion by induction. (cf. Theorem 2.1 of [H2].) But we later give another proof. \square

Proposition 2.3 says that \mathfrak{H}^1 has the commutative \mathbb{Q} -algebra structure with respect to $\bar{*}$. We denote this algebra by $\mathfrak{H}_{\bar{*}}^1$. The subset \mathfrak{H}^0 is a subalgebra of \mathfrak{H}^1 with respect to $\bar{*}$ and we denote it by $\mathfrak{H}_{\bar{*}}^0$.

We introduce the \mathbb{Q} -linear map $S : \mathfrak{H}^1 \longrightarrow \mathfrak{H}^1$. Let $S_1 \in \text{Aut}(\mathfrak{H})$ be defined by $S_1(1) = 1$, $S_1(x) = x$ and $S_1(y) = x + y$. Define the \mathbb{Q} -linear map $S : \mathfrak{H}^1 \longrightarrow \mathfrak{H}^1$ by

$$S(1) := 1 \quad \text{and} \quad S(Fy) := S_1(F)y$$

for all words $F \in \mathfrak{H}$. Then it is clear that $\bar{Z} = Z \circ S$ on \mathfrak{H}^0 , i.e.,

$$\bar{\zeta}(k_1, k_2, \dots, k_n) = Z(S(z_{k_1} z_{k_2} \cdots z_{k_n})) \quad (k_1 \geq 2).$$

For example, $\bar{\zeta}(k_1, k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) = Z(S(z_{k_1} z_{k_2}))$, $\bar{\zeta}(k_1, k_2, k_3) = \zeta(k_1 + k_2 + k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1, k_2, k_3) = Z(S(z_{k_1} z_{k_2} z_{k_3}))$. As is clear from the definition of S , we also have the following relation:

$$S(w_1 w_2) = S_1(w_1) S(w_2) \quad (w_1 \in \mathfrak{H}, w_2 \in \mathfrak{H}^1). \quad (2.1)$$

Proposition 2.4. *For $w_1, w_2 \in \mathfrak{H}^0$, we have*

$$\bar{Z}(w_1 \bar{*} w_2) = \bar{Z}(w_1) \bar{Z}(w_2).$$

To prove Proposition 2.4, we need the following lemma.

Lemma 2.5. *Let w, w_1, w_2 be words ($\neq 1$) in \mathfrak{H}^1 and p, q positive integers. Then we have*

$$S(z_p) * S_1(z_q)w = S_1(z_p z_q)w + S_1(z_q)(S(z_p) * w) - S_1(z_{p+q})w \quad (2.2)$$

and

$$\begin{aligned} S_1(z_p)w_1 * S_1(z_q)w_2 &= S_1(z_p)(w_1 * S_1(z_q)w_2) + S_1(z_q)(S_1(z_p)w_1 * w_2) \\ &\quad - S_1(z_{p+q})(w_1 * w_2) \end{aligned} \quad (2.3)$$

Proof. We first prove (2.2). Put $w = z_n \tilde{w}$ ($n \geq 1, \tilde{w} \in \mathfrak{H}^1$), then

$$\begin{aligned} \text{RHS of (2.2)} \\ = (x^{p-1}y + x^p)(x^{q-1}y + x^q)z_n \tilde{w} + (x^{q-1}y + x^q)(z_p * z_n \tilde{w}) \end{aligned}$$

$$\begin{aligned}
& - (x^{p+q-1}y + x^{p+q})z_n\tilde{w} \\
& = z_p z_q z_n \tilde{w} + z_p z_{n+q} \tilde{w} + z_{p+q} z_n \tilde{w} + z_{n+p+q} \tilde{w} \\
& \quad + z_q z_p z_n \tilde{w} + z_q z_n (z_p * \tilde{w}) + z_q z_{n+p} \tilde{w} + z_{p+q} z_n \tilde{w} + z_{n+q} (z_p * \tilde{w}) + z_{n+p+q} \tilde{w} \\
& \quad - z_{p+q} z_n \tilde{w} - z_{n+p+q} \tilde{w} \\
& = z_p * z_q z_n \tilde{w} + z_p * z_{n+q} \tilde{w} = S(z_p) * S_1(z_q) z_n \tilde{w} = S(z_p) * S_1(z_q) w.
\end{aligned}$$

Hence (2.2) follows. Putting $w_1 = z_m \tilde{w}_1$, $w_2 = z_n \tilde{w}_2$ ($m, n \geq 1$, $\tilde{w}_1, \tilde{w}_2 \in \mathfrak{H}^1$), we can prove (2.3) in the same way. \square

Proof of Proposition 2.4. It suffices to show that

$$S(w_1 \bar{*} w_2) = S(w_1) * S(w_2) \quad (2.4)$$

for words $w_1, w_2 \in \mathfrak{H}^1$. We set $w_1 = z_{p_1} z_{p_2} \cdots z_{p_m}$, $w_2 = z_{q_1} z_{q_2} \cdots z_{q_n}$. We prove (2.4) by induction on m . To ease the following calculation, we set $z_{\vec{p}} = z_{p_2} z_{p_3} \cdots z_{p_m}$ and $z_{\vec{q}} = z_{q_2} z_{q_3} \cdots z_{q_n}$. (i) We prove the case $m = 1$ by induction on n . When $n = 1$, the assertion is immediate. We assume the assertion is proven for $n - 1$. Using (2.1), (2.2) and the induction hypothesis, we have

$$\begin{aligned}
& S(z_{p_1} \bar{*} z_{q_1} z_{q_2} \cdots z_{q_n}) = S(z_{p_1} \bar{*} z_{q_1} z_{\vec{q}}) \\
& = S(z_{p_1} z_{q_1} z_{\vec{q}} + z_{q_1} (z_{p_1} \bar{*} z_{\vec{q}}) - z_{p_1+q_1} z_{\vec{q}}) \\
& = S_1(z_{p_1} z_{q_1}) S(z_{\vec{q}}) + S_1(z_{q_1}) S(z_{p_1} \bar{*} z_{\vec{q}}) - S_1(z_{p_1+q_1}) S(z_{\vec{q}}) \\
& = S_1(z_{p_1} z_{q_1}) S(z_{\vec{q}}) + S_1(z_{q_1}) (S(z_{p_1}) * S(z_{\vec{q}})) - S_1(z_{p_1+q_1}) S(z_{\vec{q}}) \\
& = S(z_{p_1}) * S_1(z_{q_1}) S(z_{\vec{q}}) \\
& = S(z_{p_1}) * S(z_{q_1} z_{\vec{q}}) = S(z_{p_1}) * S(z_{q_1} z_{q_2} \cdots z_{q_n}).
\end{aligned}$$

(ii) We assume the assertion is proven for $m - 1$. We prove the assertion for m by induction on n . When $n = 1$, the assertion follows from (i) and the commutativity of $*$, $\bar{*}$. We assume the assertion is true for $n - 1$. Using (2.1), (2.3) and the induction hypothesis, we have

$$\begin{aligned}
& S(z_{p_1} z_{p_2} \cdots z_{p_m} \bar{*} z_{q_1} z_{q_2} \cdots z_{q_n}) = S(z_{p_1} z_{\vec{p}} \bar{*} z_{q_1} z_{\vec{q}}) \\
& = S(z_{p_1} (z_{\vec{p}} \bar{*} z_{q_1} z_{\vec{q}}) + z_{q_1} (z_{p_1} z_{\vec{p}} \bar{*} z_{\vec{q}}) - z_{p_1+q_1} (z_{\vec{p}} \bar{*} z_{\vec{q}})) \\
& = S_1(z_{p_1}) S(z_{\vec{p}} \bar{*} z_{q_1} z_{\vec{q}}) + S_1(z_{q_1}) S(z_{p_1} z_{\vec{p}} \bar{*} z_{\vec{q}}) - S_1(z_{p_1+q_1}) S(z_{\vec{p}} \bar{*} z_{\vec{q}}) \\
& = S_1(z_{p_1}) (S(z_{\vec{p}}) * S(z_{q_1} z_{\vec{q}})) + S_1(z_{q_1}) (S(z_{p_1} z_{\vec{p}}) * S(z_{\vec{q}})) \\
& \quad - S_1(z_{p_1+q_1}) (S(z_{\vec{p}}) * S(z_{\vec{q}}))
\end{aligned}$$

$$\begin{aligned}
&= S_1(z_{p_1})(S(z_{\vec{p}}) * S_1(z_{q_1})S(z_{\vec{q}})) + S_1(z_{q_1})(S_1(z_{p_1})S(z_{\vec{p}}) * S(z_{\vec{q}})) \\
&\quad - S_1(z_{p_1+q_1})(S(z_{\vec{p}}) * S(z_{\vec{q}})) \\
&= S_1(z_{p_1})S(z_{\vec{p}}) * S_1(z_{q_1})S(z_{\vec{q}}) \\
&= S(z_{p_1}z_{\vec{p}}) * S(z_{q_1}z_{\vec{q}}) = S(z_{p_1}z_{p_2} \cdots z_{p_m}) * S(z_{q_1}z_{q_2} \cdots z_{q_n}).
\end{aligned}$$

This completes the proof. \square

We shall define the n -shuffle product $\overline{\text{m}}$ on \mathfrak{H} which corresponds to the shuffle product m . The n -shuffle product $\overline{\text{m}}$ is defined inductively by

$$\begin{aligned}
1 \overline{\text{m}} w &= w \overline{\text{m}} 1 = w \\
u_1 w_1 \overline{\text{m}} u_2 w_2 &= u_1(w_1 \overline{\text{m}} u_2 w_2) + u_2(u_1 w_1 \overline{\text{m}} w_2) \\
&\quad - \delta(w_1)\tau(u_1)u_2 w_2 - \delta(w_2)\tau(u_2)u_1 w_1
\end{aligned}$$

($u_1, u_2 \in \{x, y\}$ and w, w_1, w_2 are words in \mathfrak{H}), together with \mathbb{Q} -bilinearity, where δ is defined by

$$\delta(w) = \begin{cases} 1 & (w = 1), \\ 0 & (w \neq 1) \end{cases}$$

for word w and τ is defined by $\tau(x) = y, \tau(y) = x$. The n -shuffle product has the following properties.

Proposition 2.6. *The n -shuffle product is commutative and associative.*

Proof. Let w_1, w_2, w_3 be words in \mathfrak{H} . We can check the commutativity $w_1 \overline{\text{m}} w_2 = w_2 \overline{\text{m}} w_1$ by induction on $|w_1| + |w_2|$. We shall prove the associativity $(w_1 \overline{\text{m}} w_2) \overline{\text{m}} w_3 = w_1 \overline{\text{m}} (w_2 \overline{\text{m}} w_3)$ by induction on $|w_1| + |w_2| + |w_3|$. The case $|w_1| + |w_2| + |w_3| \leq 2$ is obvious. Putting $w_1 = u_1 \tilde{w}_1, w_2 = u_2 \tilde{w}_2, w_3 = u_3 \tilde{w}_3$ ($u_1, u_2, u_3 \in \{x, y\}$), we have

$$\begin{aligned}
&(w_1 \overline{\text{m}} w_2) \overline{\text{m}} w_3 \\
&= u_1(\tilde{w}_1 \overline{\text{m}} u_2 \tilde{w}_2) \overline{\text{m}} u_3 \tilde{w}_3 + u_2(u_1 \tilde{w}_1 \overline{\text{m}} \tilde{w}_2) \overline{\text{m}} u_3 \tilde{w}_3 \\
&\quad - \delta(\tilde{w}_1)\tau(u_1)u_2 \tilde{w}_2 \overline{\text{m}} u_3 \tilde{w}_3 - \delta(\tilde{w}_2)\tau(u_2)u_1 \tilde{w}_1 \overline{\text{m}} u_3 \tilde{w}_3 \\
&= u_1 \{(\tilde{w}_1 \overline{\text{m}} u_2 \tilde{w}_2) \overline{\text{m}} u_3 \tilde{w}_3\} + u_3 \{u_1(\tilde{w}_1 \overline{\text{m}} u_2 \tilde{w}_2) \overline{\text{m}} \tilde{w}_3\} \\
&\quad - \delta(\tilde{w}_3)\tau(u_3)u_1(\tilde{w}_1 \overline{\text{m}} u_2 \tilde{w}_2) + u_2 \{(u_1 \tilde{w}_1 \overline{\text{m}} \tilde{w}_2) \overline{\text{m}} u_3 \tilde{w}_3\} \\
&\quad + u_3 \{u_2(u_1 \tilde{w}_1 \overline{\text{m}} \tilde{w}_2) \overline{\text{m}} \tilde{w}_3\} - \delta(\tilde{w}_3)\tau(u_3)u_2(u_1 \tilde{w}_1 \overline{\text{m}} \tilde{w}_2) \\
&\quad - \delta(\tilde{w}_1)\tau(u_1)u_2 \tilde{w}_2 \overline{\text{m}} u_3 \tilde{w}_3 - \delta(\tilde{w}_2)\tau(u_2)u_1 \tilde{w}_1 \overline{\text{m}} u_3 \tilde{w}_3
\end{aligned}$$

$$\begin{aligned}
&= u_1 \{(\tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2) \overline{\mid} u_3 \tilde{w}_3\} + u_2 \{(u_1 \tilde{w}_1 \overline{\mid} \tilde{w}_2) \overline{\mid} u_3 \tilde{w}_3\} \\
&\quad + u_3 \{(u_1 \tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2) \overline{\mid} \tilde{w}_3\} - \delta(\tilde{w}_1) \tau(u_1) (u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \\
&\quad - \delta(\tilde{w}_2) \tau(u_2) (u_1 \tilde{w}_1 \overline{\mid} u_3 \tilde{w}_3) - \delta(\tilde{w}_3) \tau(u_3) (u_1 \tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2).
\end{aligned}$$

In the last equality, we use the following three relations:

$$\begin{aligned}
&u_1(\tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2) + u_2(u_1 \tilde{w}_1 \overline{\mid} \tilde{w}_2) \\
&\quad = u_1 \tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2 + \delta(\tilde{w}_1) \tau(u_1) u_2 \tilde{w}_2 + \delta(\tilde{w}_2) \tau(u_2) u_1 \tilde{w}_1, \\
&\tau(u_1) u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3 \\
&\quad = \tau(u_1) (u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) + u_3 (\tau(u_1) u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) - \delta(\tilde{w}_3) \tau(u_3) \tau(u_1) u_2 \tilde{w}_2, \\
&\tau(u_2) u_1 \tilde{w}_1 \overline{\mid} u_3 \tilde{w}_3 \\
&\quad = \tau(u_2) (u_1 \tilde{w}_1 \overline{\mid} u_3 \tilde{w}_3) + u_3 (\tau(u_2) u_1 \tilde{w}_1 \overline{\mid} \tilde{w}_3) - \delta(\tilde{w}_3) \tau(u_3) \tau(u_2) u_1 \tilde{w}_1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&w_1 \overline{\mid} (w_2 \overline{\mid} w_3) \\
&\quad = u_1 \tilde{w}_1 \overline{\mid} u_2 (\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) + u_1 \tilde{w}_1 \overline{\mid} u_3 (u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \\
&\quad \quad - \delta(\tilde{w}_2) u_1 \tilde{w}_1 \overline{\mid} \tau(u_2) u_3 \tilde{w}_3 - \delta(\tilde{w}_3) u_1 \tilde{w}_1 \overline{\mid} \tau(u_3) u_2 \tilde{w}_2 \\
&\quad = u_1 \{ \tilde{w}_1 \overline{\mid} u_2 (\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \} + u_2 \{ u_1 \tilde{w}_1 \overline{\mid} (\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \} \\
&\quad \quad - \delta(\tilde{w}_1) \tau(u_1) u_2 (\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) + u_1 \{ \tilde{w}_1 \overline{\mid} u_3 (u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \} \\
&\quad \quad + u_3 \{ u_1 \tilde{w}_1 \overline{\mid} (u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \} - \delta(\tilde{w}_1) \tau(u_1) u_3 (u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \\
&\quad \quad - \delta(\tilde{w}_2) u_1 \tilde{w}_1 \overline{\mid} \tau(u_2) u_3 \tilde{w}_3 - \delta(\tilde{w}_3) u_1 \tilde{w}_1 \overline{\mid} \tau(u_3) u_2 \tilde{w}_2 \\
&\quad = u_1 \{ \tilde{w}_1 \overline{\mid} (u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \} + u_2 \{ u_1 \tilde{w}_1 \overline{\mid} (\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \} \\
&\quad \quad + u_3 \{ u_1 \tilde{w}_1 \overline{\mid} (u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \} - \delta(\tilde{w}_1) \tau(u_1) (u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) \\
&\quad \quad - \delta(\tilde{w}_2) \tau(u_2) (u_1 \tilde{w}_1 \overline{\mid} u_3 \tilde{w}_3) - \delta(\tilde{w}_3) \tau(u_3) (u_1 \tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2).
\end{aligned}$$

In the last equality, we use the following three relations:

$$\begin{aligned}
&u_2(\tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3) + u_3(u_2 \tilde{w}_2 \overline{\mid} \tilde{w}_3) \\
&\quad = u_2 \tilde{w}_2 \overline{\mid} u_3 \tilde{w}_3 + \delta(\tilde{w}_2) \tau(u_2) u_3 \tilde{w}_3 + \delta(\tilde{w}_3) \tau(u_3) u_2 \tilde{w}_2, \\
&u_1 \tilde{w}_1 \overline{\mid} \tau(u_2) u_3 \tilde{w}_3 \\
&\quad = u_1 (\tilde{w}_1 \overline{\mid} \tau(u_2) u_3 \tilde{w}_3) + \tau(u_2) (u_1 \tilde{w}_1 \overline{\mid} u_3 \tilde{w}_3) - \delta(\tilde{w}_1) \tau(u_1) \tau(u_2) u_3 \tilde{w}_3, \\
&u_1 \tilde{w}_1 \overline{\mid} \tau(u_3) u_2 \tilde{w}_2 \\
&\quad = u_1 (\tilde{w}_1 \overline{\mid} \tau(u_3) u_2 \tilde{w}_2) + \tau(u_3) (u_1 \tilde{w}_1 \overline{\mid} u_2 \tilde{w}_2) - \delta(\tilde{w}_1) \tau(u_1) \tau(u_3) u_2 \tilde{w}_2.
\end{aligned}$$

So we have the assertion by the induction hypothesis. \square

Proposition 2.6 says that \mathfrak{H} has the commutative \mathbb{Q} -algebra structure with respect to $\overline{\mathfrak{m}}$. We denote it by $\mathfrak{H}_{\overline{\mathfrak{m}}}$. Subsets \mathfrak{H}^1 and \mathfrak{H}^0 are subalgebras of \mathfrak{H} with respect to $\overline{\mathfrak{m}}$ and we denote them by $\mathfrak{H}_{\overline{\mathfrak{m}}}^1$, $\mathfrak{H}_{\overline{\mathfrak{m}}}^0$ respectively.

Proposition 2.7. *For $w_1, w_2 \in \mathfrak{H}^0$, we have*

$$\overline{Z}(w_1 \overline{\mathfrak{m}} w_2) = \overline{Z}(w_1) \overline{Z}(w_2).$$

Proof. It suffices to prove that

$$S(w_1 \overline{\mathfrak{m}} w_2) = S(w_1) \mathfrak{m} S(w_2) \quad (2.5)$$

for words $w_1, w_2 \in \mathfrak{H}^1$. We put $w_1 = u_1 u_2 \cdots u_m$ and $w_2 = v_1 v_2 \cdots v_n$ ($u_i, v_i \in \{x, y\}$). We prove (2.5) by induction on m . In order to simplify the proof, we set $u_{\vec{m}} := u_2 u_3 \cdots u_m$ and $v_{\vec{n}} := v_2 v_3 \cdots v_n$. (i) We prove the case $m = 1$ by induction on n .

$$\begin{aligned} S(u_1 \overline{\mathfrak{m}} v_1) &= S(y \overline{\mathfrak{m}} y) = S(2y^2 - 2xy) = 2(x + y)y - 2xy = 2y^2 \\ &= y \mathfrak{m} y = S(y) \mathfrak{m} S(y) = S(u_1) \mathfrak{m} S(v_1). \end{aligned}$$

So the case $n = 1$ is valid. We assume the assertion is proven for $n - 1$. Using (2.1) and the induction hypothesis, we have

$$\begin{aligned} S(u_1 \overline{\mathfrak{m}} v_1 v_2 \cdots v_n) &= S(y \overline{\mathfrak{m}} v_1 v_{\vec{n}}) \\ &= S(y v_1 v_{\vec{n}} + v_1(y \overline{\mathfrak{m}} v_{\vec{n}}) - x v_1 v_{\vec{n}}) \\ &= S_1(y) S_1(v_1) S(v_{\vec{n}}) + S_1(v_1) S(y \overline{\mathfrak{m}} v_{\vec{n}}) - S_1(x) S_1(v_1) S(v_{\vec{n}}) \\ &= (x + y) S_1(v_1) S(v_{\vec{n}}) + S_1(v_1) (S(y) \mathfrak{m} S(v_{\vec{n}})) - x S_1(v_1) S(v_{\vec{n}}) \\ &= y S_1(v_1) S(v_{\vec{n}}) + S_1(v_1) (y \mathfrak{m} S(v_{\vec{n}})) \\ &= y \mathfrak{m} S_1(v_1) S(v_{\vec{n}}) \\ &= S(u_1) \mathfrak{m} S(v_1 v_{\vec{n}}) = S(u_1) \mathfrak{m} S(v_1 v_2 \cdots v_n). \end{aligned}$$

Therefore we have the assertion for n . (ii) We assume the assertion is proven for $m - 1$. We prove the assertion for m by induction on n . The case $n = 1$ is obvious by (i) and the commutativity of \mathfrak{m} , $\overline{\mathfrak{m}}$. We assume the assertion is true for $n - 1$.

$$\begin{aligned} S(u_1 u_2 \cdots u_m \overline{\mathfrak{m}} v_1 v_2 \cdots v_n) &= S(u_1 u_{\vec{m}} \overline{\mathfrak{m}} v_1 v_{\vec{n}}) \\ &= S(u_1 (u_{\vec{m}} \overline{\mathfrak{m}} v_1 v_{\vec{n}}) + v_1 (u_1 u_{\vec{m}} \overline{\mathfrak{m}} v_{\vec{n}})) \end{aligned}$$

$$\begin{aligned}
&= S_1(u_1)S(u_{\vec{m}} \overline{\text{III}} v_1 v_{\vec{n}}) + S_1(v_1)S(u_1 u_{\vec{m}} \overline{\text{III}} v_{\vec{n}}) \\
&= S_1(u_1)(S(u_{\vec{m}}) \text{III} S(v_1 v_{\vec{n}})) + S_1(v_1)(S(u_1 u_{\vec{m}}) \text{III} S(v_{\vec{n}})) \\
&= S_1(u_1)(S(u_{\vec{m}}) \text{III} S_1(v_1)S(v_{\vec{n}})) + S_1(v_1)(S_1(u_1)S(u_{\vec{m}}) \text{III} S(v_{\vec{n}})) \\
&= S_1(u_1)S(u_{\vec{m}}) \text{III} S_1(v_1)S(v_{\vec{n}}) \\
&= S(u_1 u_{\vec{m}}) \text{III} S(v_1 v_{\vec{n}}) = S(u_1 u_2 \cdots u_m) \text{III} S(v_1 v_2 \cdots v_n).
\end{aligned}$$

This completes the proof. \square

Because the n-evaluation map \overline{Z} is homomorphism with respect to $\overline{\ast}$ and $\overline{\text{III}}$, we have the following theorem.

Theorem 2.8 (Finite double shuffle relations of NMZVs). *For $w_1, w_2 \in \mathfrak{H}^0$, we have*

$$\overline{Z}(w_1 \overline{\ast} w_2 - w_1 \overline{\text{III}} w_2) = 0.$$

2.3 Extended double shuffle relations of NMZVs

In this subsection, we generalize Theorem 2.8. In the following lemma, we introduce the inverse of S .

Lemma 2.9. (i) *Let $S_2 \in \mathfrak{H}$ be defined by $S_2(1) = 1$, $S_2(x) = x$ and $S_2(y) = y - x$. And we define \mathbb{Q} -linear map $\tilde{S} : \mathfrak{H}^1 \longrightarrow \mathfrak{H}^1$ by*

$$\tilde{S}(1) = 1 \quad \text{and} \quad \tilde{S}(Fy) := S_2(F)y$$

for words $F \in \mathfrak{H}$. Then we have $\tilde{S} \circ S = S \circ \tilde{S} = \text{id}$ on \mathfrak{H}^1 .

(ii) *For $w_1, w_2 \in \mathfrak{H}^1$, we have*

$$\begin{aligned}
\tilde{S}(w_1 \ast w_2) &= \tilde{S}(w_1) \overline{\ast} \tilde{S}(w_2), \\
\tilde{S}(w_1 \text{III} w_2) &= \tilde{S}(w_1) \overline{\text{III}} \tilde{S}(w_2).
\end{aligned}$$

Proof. (i) By definition, we have $\tilde{S} \circ S(1) = S \circ \tilde{S}(1) = 1$. Let w be contained in $\mathfrak{H}^1 \setminus \{1\}$. Then we can write $w = w_1 y$ ($w_1 \in \mathfrak{H}$), and we have

$$\tilde{S} \circ S(w) = \tilde{S} \circ S(w_1 y) = \tilde{S}(S_1(w_1)y) = S_2(S_1(w_1))y = w_1 y = w,$$

$$S \circ \tilde{S}(w) = S \circ \tilde{S}(w_1 y) = S(S_2(w_1)y) = S_1(S_2(w_1))y = w_1 y = w.$$

This completes the proof of (i). (ii) This is clear from (2.4), (2.5) and (i). \square

By (i) of Lemma 2.9, we can rewrite \tilde{S} by S^{-1} . Then (ii) of Lemma 2.9 can be restated as follows:

$$S^{-1}(w_1 * w_2) = S^{-1}(w_1) \bar{*} S^{-1}(w_2), \quad (2.6)$$

$$S^{-1}(w_1 \text{III} w_2) = S^{-1}(w_1) \overline{\text{III}} S^{-1}(w_2). \quad (2.7)$$

Using (2.6), we give the proof of Proposition 2.3

Proof of Proposition 2.3. By using (2.6) and the commutativity of the harmonic product $*$, we have

$$\begin{aligned} w_1 \bar{*} w_2 &= S^{-1}(S(w_1)) \bar{*} S^{-1}(S(w_2)) = S^{-1}(S(w_1) * S(w_2)) \\ &= S^{-1}(S(w_2) * S(w_1)) = S^{-1}(S(w_2)) \bar{*} S^{-1}(S(w_1)) = w_2 \bar{*} w_1. \end{aligned}$$

So the commutativity of n-harmonic product $\bar{*}$ follows. We next prove the associativity of n-harmonic product $\bar{*}$ by using (2.6) and the associativity of the harmonic product $*$.

$$\begin{aligned} w_1 \bar{*} (w_2 \bar{*} w_3) &= S^{-1}(S(w_1)) \bar{*} (S^{-1}(S(w_2)) \bar{*} S^{-1}(S(w_3))) \\ &= S^{-1}(S(w_1)) \bar{*} S^{-1}(S(w_2) * S(w_3)) \\ &= S^{-1}(S(w_1) * (S(w_2) * S(w_3))) \\ &= S^{-1}((S(w_1) * S(w_2)) * S(w_3)) \\ &= S^{-1}(S(w_1) * S(w_2)) \bar{*} S^{-1}(S(w_3)) \\ &= (S^{-1}(S(w_1)) \bar{*} S^{-1}(S(w_2))) \bar{*} w_3 \\ &= (w_1 \bar{*} w_2) \bar{*} w_3. \end{aligned}$$

This completes the proof. \square

Lemma 2.10. *Let $\circ = *$ or III . A word $y^m w$ ($m \geq 0, w \in \mathfrak{H}^0$) of \mathfrak{H}^1 is represented uniquely by*

$$y^m w = w_0 + w_1 \bar{\circ} y + w_2 \bar{\circ} y^{\bar{\circ} 2} + \cdots + w_m \bar{\circ} y^{\bar{\circ} m} \quad (w_i \in \mathfrak{H}^0), \quad (2.8)$$

i.e., we have $\mathfrak{H}_{\bar{\circ}}^0[y] \simeq \mathfrak{H}_{\bar{\circ}}^1$.

Proof. We first prove that $y^m w$ can be represented as (2.8). By Corollary 5 of [IKZ], we have

$$(y+x)^m S(w) = \sum_{i=0}^m v_i \circ y^{\circ i} \quad (v_i \in \mathfrak{H}^0).$$

Using (2.6) or (2.7), we obtain

$$y^m w = \sum_{i=0}^m S^{-1}(v_i) \bar{\circ} y^{\bar{\circ} i}.$$

(We have $S^{-1}(w_1 w_2) = S_2(w_1) S^{-1}(w_2)$ for $w_1 \in \mathfrak{H}$, $w_2 \in \mathfrak{H}^1$.) Therefore, the first assertion follows from $S^{-1}(\mathfrak{H}^0) \subset \mathfrak{H}^0$. We next prove the uniqueness of representation (2.8). We put

$$\sum_{i=0}^m w_i \bar{\circ} y^{\bar{\circ} i} = \sum_{i=0}^m v_i \bar{\circ} y^{\bar{\circ} i} \quad (w_i, v_i \in \mathfrak{H}^0).$$

Using (2.4) or (2.5), we have

$$\sum_{i=0}^m S(w_i) \circ y^{\circ i} = \sum_{i=0}^m S(v_i) \circ y^{\circ i}.$$

By $\mathfrak{H}_\circ^0[y] \simeq \mathfrak{H}_\circ^1$ (see [H2] and [R]), we have $S(w_i) = S(v_i)$ for $i = 0, 1, \dots, m$. So we have the second assertion. \square

Proposition 2.11. *We have two algebra homomorphisms*

$$\bar{Z}^{\bar{*}} : \mathfrak{H}_{\bar{*}}^1 \longrightarrow \mathbb{R}[T] \quad \text{and} \quad \bar{Z}^{\bar{\text{III}}} : \mathfrak{H}_{\bar{\text{III}}}^1 \longrightarrow \mathbb{R}[T]$$

which are uniquely characterized by the properties that they both extend the n -evaluation map $\bar{Z} : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ and send y to T .

Proof. The assertion follows because \bar{Z} is homomorphism respect to $\bar{*}$, $\bar{\text{III}}$ and we have isomorphisms $\mathfrak{H}_{\bar{*}}^0[y] \simeq \mathfrak{H}_{\bar{*}}^1$, $\mathfrak{H}_{\bar{\text{III}}}^0[y] \simeq \mathfrak{H}_{\bar{\text{III}}}^1$. \square

The \mathbb{Q} -algebra homomorphisms $\bar{Z}^{\bar{*}}$, $\bar{Z}^{\bar{\text{III}}}$ have the following relations:

$$\bar{Z}^{\bar{*}} = Z^* \circ S, \quad \bar{Z}^{\bar{\text{III}}} = Z^{\text{III}} \circ S.$$

(\circ means composition) Indeed, $Z^* \circ S$ and $Z^{\text{III}} \circ S$ satisfy the conditions of Proposition 2.11.

Theorem 2.12 (Extended double shuffle relations of NMZVs). *For $w_1 \in \mathfrak{H}^1$ and $w_2 \in \mathfrak{H}^0$, we have*

$$\overline{Z}^*(w_1 \overline{\boxplus} w_2 - w_1 \overline{*} w_2) = 0 \quad \text{and} \quad \overline{Z}^{\overline{\boxplus}}(w_1 \overline{\boxplus} w_2 - w_1 \overline{*} w_2) = 0.$$

Proof. By using (2.4), (2.5) and the relation $\overline{Z}^* = Z^* \circ S$, we have

$$\begin{aligned} \overline{Z}^*(w_1 \overline{\boxplus} w_0 - w_1 \overline{*} w_0) &= Z^* \circ S(w_1 \overline{\boxplus} w_0 - w_1 \overline{*} w_0) \\ &= Z^* \left(S(w_1) \boxplus S(w_0) - S(w_1) * S(w_0) \right) \\ &= 0. \end{aligned}$$

The last equality follows by Theorem 2.2 and the fact $S(\mathfrak{H}^1) \subset \mathfrak{H}^1$, $S(\mathfrak{H}^0) \subset \mathfrak{H}^0$. The other identity can be proven in the same way. \square

3 Application

In [H1], Hoffman proved the following theorem.

Theorem 3.1 ([H1]). *For positive integers k_1, k_2, \dots, k_n and $k_1 \geq 2$, we have*

$$\begin{aligned} \sum_{i=1}^n \zeta(k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_n) \\ = \sum_{\substack{1 \leq i \leq n \\ k_i \geq 2}} \sum_{j=0}^{k_i-2} \zeta(k_1, \dots, k_{i-1}, k_i - j, j + 1, k_{i+1}, \dots, k_n). \end{aligned}$$

In this section, we prove an analogue of Hoffman's relations for NMZVs:

Theorem 3.2. *For positive integers k_1, k_2, \dots, k_n and $k_1 \geq 2$, we have*

$$\begin{aligned} \sum_{i=1}^n (k_i - 1 + \delta_{ni}) \overline{\zeta}(k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_n) \\ = \sum_{\substack{1 \leq i \leq n \\ k_i \geq 2}} \sum_{j=0}^{k_i-2} \overline{\zeta}(k_1, \dots, k_{i-1}, k_i - j, j + 1, k_{i+1}, \dots, k_n). \end{aligned}$$

We first prove the following lemma.

Lemma 3.3. (i) Let w be a word in \mathfrak{H} and let k a positive integer. Then we have

$$y \overline{\equiv} z_k w = \begin{cases} z_1 z_k + \sum_{j=0}^{k-2} z_{k-j} z_{j+1} - (k+1) z_{k+1} + z_k z_1 & (w = 1), \\ z_1 z_k w + \sum_{j=0}^{k-2} z_{k-j} z_{j+1} w - k z_{k+1} w + z_k (y \overline{\equiv} w) & (w \neq 1), \end{cases}$$

where the summation is treated as 0 when $k = 1$.

(ii) For $k_1, k_2, \dots, k_n \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned} y \overline{\equiv} z_{k_1} z_{k_2} \cdots z_{k_n} &= \sum_{i=0}^n z_{k_1} \cdots z_{k_i} z_1 z_{k_{i+1}} \cdots z_{k_n} \\ &+ \sum_{\substack{1 \leq i \leq n \\ k_i \geq 2}} \sum_{j=0}^{k_i-2} z_{k_1} \cdots z_{k_{i-1}} z_{k_i-j} z_{j+1} z_{k_{i+1}} \cdots z_{k_n} \\ &- \sum_{i=1}^n (k_i + \delta_{ni}) z_{k_1} \cdots z_{k_{i-1}} z_{k_{i+1}} z_{k_{i+1}} \cdots z_{k_n}. \end{aligned} \quad (3.1)$$

Proof. (i) The case $k = 1$ is clear from the definition of $\overline{\equiv}$. And we can prove the case $k \geq 2$ by induction on k . (ii) We prove the assertion by induction on n . The case $n = 1$ follows from (i). We assume that the assertion is true for $n - 1$. Using (i), we obtain

$$\begin{aligned} y \overline{\equiv} z_{k_1} z_{k_2} \cdots z_{k_n} &= z_1 z_{k_1} z_{k_2} \cdots z_{k_n} + \sum_{j=0}^{k_1-2} z_{k_1-j} z_{j+1} z_{k_2} \cdots z_{k_n} \\ &- k_1 z_{k_1+1} z_{k_2} \cdots z_{k_n} + z_{k_1} (y \overline{\equiv} z_{k_2} \cdots z_{k_n}). \end{aligned}$$

By the induction hypothesis, this expression equals the right hand side of (3.1). \square

Proof of Theorem 3.2. By Lemma 3.3 and the definition of $\overline{*}$, we have

$$\begin{aligned}
& y \overline{\text{III}} z_{k_1} z_{k_2} \cdots z_{k_n} - y \overline{*} z_{k_1} z_{k_2} \cdots z_{k_n} \\
&= \sum_{\substack{1 \leq i \leq n \\ k_i \geq 2}} \sum_{j=0}^{k_i-2} z_{k_1} \cdots z_{k_{i-1}} z_{k_i-j} z_{j+1} z_{k_{i+1}} \cdots z_{k_n} \\
&\quad - \sum_{i=1}^n (k_i + \delta_{ni} - 1) z_{k_1} \cdots z_{k_{i-1}} z_{k_i+1} z_{k_{i+1}} \cdots z_{k_n}.
\end{aligned}$$

The right hand side is contained in \mathfrak{H}^0 by $k_1 \geq 2$. Therefore, the assertion follows from Theorem 2.12. \square

References

- [AO] T. Aoki and Y. Ohno, *Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric function*, Publ. Res. Inst. Math. Sci. **41** (2005), 329–337.
- [IKZ] K. Ihara, M. Kaneko, D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. **142** (2006), 307–338.
- [H1] M. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), 275–290.
- [H2] M. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495.
- [HO] M. Hoffman and Y. Ohno, *Relations of multiple zeta values and their algebraic expression*, J. Algebra **262** (2003), 332–347.
- [O] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory **74** (1999), 39–43.
- [OW] Y. Ohno and N. Wakabayashi, *Cyclic sum of multiple zeta values*, Acta Arith. **123** (2006), 289–295.
- [R] C. Reutenauer, *Free Lie Algebras* (Oxford Science Publications, Oxford, 1993).

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY
FUKUOKA 812-8581, JAPAN
E-mail address: muneta@math.kyushu-u.ac.jp